CO-STABILITY OF RADICALS AND ITS APPLICATIONS TO PI-THEORY

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ABSTRACT. We prove that if A is a finite dimensional associative H-comodule algebra over a field F for some involutory Hopf algebra H not necessarily finite dimensional, where either $\operatorname{char} F = 0$ or $\operatorname{char} F > \dim A$, then the Jacobson radical $J(A)$ is an H-subcomodule of A. In particular, if A is a finite dimensional associative algebra over such a field F, graded by any group, then the Jacobson radical $J(A)$ is a graded ideal of A. Analogous results hold for nilpotent and solvable radicals of finite dimensional Lie algebras over a field of characteristic 0. We use the results obtained to prove the analog of Amitsur's conjecture for graded polynomial identities of finite dimensional associative algebras over a field of characteristic 0, graded by any group.

1. INTRODUCTION

If an algebra is endowed with an additional structure, e.g. grading, action of a group, Lie or Hopf algebra, a natural question arises whether the radical is invariant with respect to this structure.

In 1984 M. Cohen and S. Montgomery [7] proved that the Jacobson radical of an associative algebra graded by a finite group G is graded if $|G|^{-1}$ belongs to the base field. In 2001 V. Linchenko [15] proved the stability of the Jacobson radical of a finite dimensional H-module associative algebra for an involutory Hopf algebra H. This result was later generalized by V. Linchenko, S. Montgomery, L.W. Small, and S.M. Skryabin [16, 19].

In 2011, D. Pagon, D. Repovš, and M.V. Zaicev [18, Proposition 3.3 and its proof] proved that the solvable radical of a finite dimensional Lie algebra over an algebraically closed field of characteristic 0, graded by any group, is graded. In 2012, the author [11, Theorem 1] proved that the solvable and the nilpotent radical of a finite dimensional H-(co)module Lie algebra over a field of characteristic 0 are H-(co)invariant for any finite dimensional (co)semisimple Hopf algebra H. (In fact, exactly the same arguments can be used to derive the stability of the radicals in any finite dimensional H-module Lie algebra for an arbitrary involutory Hopf algebra H not necessarily finite dimensional.)

Here we prove the co-stability of the Jacobson radical of a finite dimensional H-comodule associative algebra A for an involutory Hopf algebra H under some restrictions on the characteristic of the base field (Theorem 1). Unfortunately, if H is infinite dimensional, we cannot derive this result from V. Linchenko's one directly since the dual algebra H^* is not necessarily a coalgebra. However, using the fact that A is finite dimensional, we provide a substitute for a comultiplication in H^* (see Lemma 1), which is enough for our purposes.

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As a consequence, we prove that the nilpotent and the solvable radical of a finite dimensional H-comodule Lie algebra for an involutory Hopf algebra H over a field of characteristic 0, are H-subcomodules (Theorem 2).

As another consequence, we show that the radicals of finite dimensional algebras graded by an arbitrary group are graded. The latter result is used to prove the analog of Amitsur's conjecture for graded codimensions of finite dimensional algebras.

The original Amitsur's conjecture was proved in 1999 by A. Giambruno and M.V. Zaicev [10, Theorem 6.5.2] for all associative PI-algebras. Alongside with ordinary polynomial identities of algebras, graded, differential, G- and H-identities are important too [4, 5, 6]. Usually, to find such identities is easier than to find the ordinary ones. Furthermore, each of these types of identities completely determines the ordinary polynomial identities. Therefore the question arises whether the conjecture holds for graded codimensions, G-, H-codimensions, and codimensions of polynomial identities with derivations. The analog of Amitsur's conjecture for codimensions of graded identities was proved in 2010–2011 by E. Aljadeff, A. Giambruno, and D. La Mattina [2, 3, 9] for all associative PI-algebras graded by a finite group. In 2012, the author and M.V Kotchetov [13, Theorem 13] proved the analog of Amitsur's conjecture for graded polynomial identities of finite dimensional associative algebras graded by an Abelian group.

Here we prove this conjecture for finite dimensional associative algebras graded by an arbitrary group (Theorem 3).

2. H-comodule algebras

Let H be a Hopf algebra over a field F with a comultiplication $\Delta\colon H \to H \otimes H$, a counit $\varepsilon\colon H \to F$, and an antipode $S\colon H \to H$. We say that H is *involutory* if $S^2 = \mathrm{id}_H$. Throughout the paper we use Sweedler's notation $\Delta h = h_{(1)} \otimes h_{(2)}$ where Δ is the comultiplication in H. We refer the reader to [8, 17, 21] for an account of Hopf algebras and algebras with Hopf algebra actions.

Recall that an algebra A is a *(right) H-comodule algebra* if A is a right H-comodule and $\rho(ab) = a_{(0)}b_{(0)} \otimes a_{(1)}b_{(1)}$ for all $a, b \in A$ where $\rho\colon A \to A \otimes H$ is the comodule map. (Here we use Sweedler's notation $\rho(a) = a_{(0)} \otimes a_{(1)}$ for $a \in A$.)

Example 1. Let $A = \bigoplus_{g\in G} A^{(g)}$ be a graded algebra for some group G. Then A is an FG-comodule algebra for the group algebra FG where $\rho(a^{(g)}) = a^{(g)} \otimes g$ for all $a^{(g)} \in A^{(g)}$, $g \in G$.

Note that an H-comodule algebra A is a left H^*-module where H^* is the algebra dual to the coalgebra H and $h^*a := h^*(a_{(1)})a_{(0)}$ for all $a \in A$ and $h^* \in H^*$. If H is infinite dimensional, then it is not always possible to define the structure of a coalgebra on H^* dual to the algebra H. However, we can provide some substitute for a comultiplication in H^*.

Lemma 1. *Suppose A is a finite dimensional H-comodule algebra for some Hopf algebra H over a field F. Choose a finite dimensional subspace H_1 such that $\rho(A) \subseteq A \otimes H_1$. Then for every $h^* \in H^*$ there exist $s \in \mathbb{N}$, $h^{*\prime}_i, h^{*\prime\prime}_i \in H^*$, $1 \leqslant i \leqslant s$, such that*

$$h^*(hq) = \sum_{i=1}^{s} h^{*\prime}_i(h)h^{*\prime\prime}_i(q) \text{ for all } h, q \in H_1.$$

In particular, $h^*(ab) = \sum_{i=1}^{s}(h^{*\prime}_i a)(h^{*\prime\prime}_i b)$ for all $a, b \in A$.

Proof. Consider the map $\Xi\colon H^* \to (H_1 \otimes H_1)^*$ where $\Xi(h^*)(h \otimes q) = h^*(hq)$ for $h^* \in H^*$, $h, q \in H$. Using the natural identification $(H_1 \otimes H_1)^* = H_1^* \otimes H_1^*$, we get $\Xi(h^*) = \sum_{i=1}^{s} h^{*\prime}_i \otimes h^{*\prime\prime}_i$

for some $s \in \mathbb{N}$, $h_i^{*\prime}, h_i^{*\prime\prime} \in H_1^*$, $1 \leqslant i \leqslant s$. Extending linear functions from H_1 to H, we may assume that $h_i^{*\prime}, h_i^{*\prime\prime} \in H^*$. Then $h^*(hq) = \sum_{i=1}^s h_i^{*\prime}(h)h_i^{*\prime\prime}(q)$ for all $h, q \in H_1$. In particular,

$$h^*(ab) = h^*(a_{(1)}b_{(1)})a_{(0)}b_{(0)} = \sum_{i=1}^s h_i^{*\prime}(a_{(1)})h_i^{*\prime\prime}(b_{(1)})a_{(0)}b_{(0)} = \sum_{i=1}^s (h_i^{*\prime}a)(h_i^{*\prime\prime}b).$$

for all $a, b \in A$. $\square$

Remark. Throughout the paper we choose H_1 as follows. First, we choose a finite dimensional subspace $H_2 \subseteq H$ such that $\rho(A) \subseteq A \otimes H_2$. Second, we choose a finite dimensional subspace $H_2 \subseteq H_3 \subseteq H$ such that $\Delta(H_2) \subseteq H_3 \otimes H_3$. Finally, we define $H_1 := H_3 + SH_3$. Now we may assume that $a_{(1)}, a_{(2)}, Sa_{(1)}, Sa_{(2)} \in H_1$ for all $a \in A$ in expressions like

$$(\mathrm{id}_A \otimes \Delta)\rho(a) = a_{(0)} \otimes a_{(1)} \otimes a_{(2)}.$$

Lemma 2. *Let A be a finite dimensional H-comodule algebra over a field F for some Hopf algebra H with a bijective antipode. If I is an ideal of A, then H^*I is an ideal of A too.*

Proof. Let $a \in I$, $b \in A$, $h^* \in H^*$. Then

$$\begin{aligned}(h^*a)b = h^*(a_{(1)})a_{(0)}b = h^*(\varepsilon(b_{(1)})a_{(1)})a_{(0)}b_{(0)} = h^*(a_{(1)}b_{(1)}Sb_{(2)})a_{(0)}b_{(0)} = \\ \sum_i h_i^{*\prime}(a_{(1)}b_{(1)})h_i^{*\prime\prime}(Sb_{(2)})a_{(0)}b_{(0)} = \sum_i h_i^{*\prime}(a(S^*h_i^{*\prime\prime})b) \in H^*I.\end{aligned} \tag{1}$$

Similarly,

$$\begin{aligned}b(h^*a) = h^*(a_{(1)})ba_{(0)} = h^*(\varepsilon(b_{(1)})a_{(1)})b_{(0)}a_{(0)} = h^*((S^{-1}b_{(2)})b_{(1)}a_{(1)})b_{(0)}a_{(0)} = \\ \sum_i h_i^{*\prime}(S^{-1}b_{(2)})h_i^{*\prime\prime}(b_{(1)}a_{(1)})b_{(0)}a_{(0)} = \sum_i h_i^{*\prime\prime}((((S^{-1})^*h_i^{*\prime})b)a) \in H^*I.\end{aligned} \tag{2}$$

$\square$

In addition, we need the Wedderburn — Artin theorem for H-comodule algebras:

Lemma 3. *Let B be a finite dimensional semisimple associative H-comodule algebra over a field F for some Hopf algebra H with a bijective antipode. Then $B = B_1 \oplus B_2 \oplus \ldots \oplus B_s$ (direct sum of H-coinvariant ideals) for some H-simple algebras B_i.*

Proof. By the original Wedderburn — Artin theorem, $B = A_1 \oplus \ldots \oplus A_s$ (direct sum of ideals) where A_i are simple algebras not necessarily H-subcomodules. Let B_1 be a minimal ideal of A that is an H-subcomodule. Then $B_1 = A_{i_1} \oplus \ldots \oplus A_{i_k}$ for some $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, s\}$. Consider $\tilde{B}_1 = \{a \in A \mid ab = ba = 0 \text{ for all } b \in B_1\}$. Then $\tilde{B}_1$ equals the sum of all A_j, $j \notin \{i_1, i_2, \ldots, i_k\}$, and $A = B_1 \oplus \tilde{B}_1$. We claim that $\tilde{B}_1$ is an H-subcomodule. It is sufficient to prove that for every $h^* \in H^*$, $a \in \tilde{B}_1$, $b \in B_1$, we have $h^*(a_{(1)})a_{(0)}b = h^*(a_{(1)})ba_{(0)} = 0$. However, this follows from (1) and (2). Hence $\tilde{B}_1$ is an H-subcomodule and the inductive argument finishes the proof. $\square$

3. Co-stability of the Jacobson radical

Note that if A is a finite dimensional H-comodule algebra for a Hopf algebra H, then $\mathrm{End}_F(A)$ is an H-comodule algebra where $\psi_{(0)}a \otimes \psi_{(1)} = \psi(a_{(0)})_{(0)} \otimes \psi(a_{(0)})_{(1)}(Sa_{(1)})$ for $\psi \in \mathrm{End}_F(A)$ and $a \in A$. Moreover,

$$(h^*\psi)a = h^*(\psi(a_{(0)})_{(1)}(Sa_{(1)}))\psi(a_{(0)})_{(0)} = \sum_i h_i^{*\prime}\psi((S^*h_i^{*\prime\prime})a)$$

for $h^* \in H^*$, $\psi \in \mathrm{End}_F(A)$, and $a \in A$. Hence $h^*\psi = \sum_i \zeta(h_i^{*\prime})\psi\zeta(S^*h_i^{*\prime\prime})$ for all $h^* \in H^*$ and $\psi \in \mathrm{End}_F(A)$ where $\zeta \colon H^* \to \mathrm{End}_F(A)$ is the map corresponding to the H^*-module structure on A.

Lemma 4. *Let A be a finite dimensional H-comodule algebra over a field F for some Hopf algebra H with $S^2 = \mathrm{id}_H$. Consider the left regular representation $\Phi\colon A \to \mathrm{End}_F(A)$ where $\Phi(a)b := ab$. Then*

$$\mathrm{tr}(\Phi(h^*a)) = h^*(1)\,\mathrm{tr}(\Phi(a)) \text{ for all } h^* \in H^* \text{ and } a \in A.$$

Proof. Note that Φ is a homomorphism of H-comodules. Therefore, Φ is a homomorphism of H^*-modules. Moreover,

$$\mathrm{tr}(\Phi(h^*a)) = \sum_i \mathrm{tr}\,\zeta(h_i^{*\prime})\Phi(a)\zeta(S^*h_i^{*\prime\prime}) = \sum_i \mathrm{tr}(\zeta((S^*h_i^{*\prime\prime})h_i^{*\prime})\Phi(a)).$$

Note that

$$\sum_i \zeta((S^*h_i^{*\prime\prime})h_i^{*\prime})b = \sum_i (S^*h_i^{*\prime\prime})(h_i^{*\prime}(b_{(1)})b_{(0)}) =$$

$$\sum_i (h_i^{*\prime\prime})(Sb_{(1)})h_i^{*\prime}(b_{(2)})b_{(0)} = h^*(b_{(2)}Sb_{(1)})b_{(0)} = h^*(1)b$$

for all $h^* \in H^*$ and $b \in A$. Hence $\mathrm{tr}(\Phi(h^*a)) = h^*(1)\,\mathrm{tr}(\Phi(a))$. $\square$

Theorem 1. *Let A be a finite dimensional associative H-comodule algebra over a field F for some Hopf algebra H with $S^2 = \mathrm{id}_H$. Suppose that either* $\mathrm{char}\, F = 0$ *or* $\mathrm{char}\, F > \dim A$. *Then the Jacobson radical $J := J(A)$ is an H-subcomodule of A.*

Corollary. *Let A be a finite dimensional associative algebra over a field F graded by any group G. Suppose that either* $\mathrm{char}\, F = 0$ *or* $\mathrm{char}\, F > \dim A$. *Then the Jacobson radical $J := J(A)$ is a graded ideal of A.*

Combining this with [20, Corollary 2.8], we get the graded Wedderburn — Mal'cev theorem:

Corollary. *Let A be a finite dimensional associative algebra over a field F graded by any group G. Suppose that either* $\mathrm{char}\, F = 0$ *or* $\mathrm{char}\, F > \dim A$ *and $A/J(A)$ is separable. Then there exists a maximal semisimple subalgebra $B \subseteq A$ such that $A = B \oplus J(A)$ (direct sum of graded subspaces).*

Proof of Theorem 1. Note that $J_0 := H^*J \supseteq J$ is an H-subcomodule. By Lemma 2, J_0 is an ideal. Therefore, it is sufficient to prove that J_0 is a nil-ideal.

Let $a_1, \ldots, a_m \in J$, $h_1^*, \ldots, h_m^* \in H^*$. Note that J is nilpotent and $\mathrm{tr}(\Phi(a)) = 0$ for all $a \in J$. By (1) and Lemma 4,

$$\mathrm{tr}\,(\Phi((h_1^*a_1)\ldots(h_m^*a_m))) = \sum_i \mathrm{tr}\left(\Phi\left(h_{1i}^{*\prime\prime}(a_1(S^*h_{1i}^{*\prime\prime})((h_2^*a_2)\ldots(h_m^*a_m)))\right)\right) =$$

$$\sum_i h_{1i}^{*\prime\prime}(1)\,\mathrm{tr}\left(\Phi\left(a_1(S^*h_{1i}^{*\prime\prime})((h_2^*a_2)\ldots(h_m^*a_m))\right)\right) = 0$$

since $a_1(S^*h_{1i}^{*\prime\prime})((h_2^*a_2)\ldots(h_m^*a_m)) \in J$.

In particular, $\mathrm{tr}(\Phi(a)^k) = 0$ for all $a \in J_0$ and $k \in \mathbb{N}$. Since either $\mathrm{char}\, F = 0$ or $\mathrm{char}\, F > \dim A$, $\Phi(a)$ is a nilpotent operator on A and J_0 is a nil-ideal. Hence $J = J_0$. $\square$

4. Co-stability of radicals in Lie algebras

Analogous results hold for Lie algebras:

Theorem 2. *Let L be a finite dimensional H-comodule Lie algebra over a field F of characteristic 0 for some Hopf algebra H with $S^2 = \mathrm{id}_H$. Then the solvable radical R and the nilpotent radical N of L are H-subcomodules.*

Proof. Consider the adjoint representation $\mathrm{ad}\colon L \to \mathfrak{gl}(L)$ where $(\mathrm{ad}\, a)b := [a,b]$. Then ad is a homomorphism of H-comodules and H^*-modules. Denote by A the associative subalgebra of $\mathrm{End}_F(L)$ generated by $(\mathrm{ad}\, L)$. Applying Lemma 1 to $\mathrm{End}_F(L)$, we obtain that A is an H^*-submodule. Therefore, A is an H-subcomodule. By Lemma 2, H^*N and H^*R are ideals of L. Theorem 1 and [11, Lemma 1] imply $(\mathrm{ad}(H^*N)) \subseteq H^*J(A) = J(A)$. Thus H^*N is nilpotent and $H^*N = N$.

By [14, Proposition 2.1.7], $[L,R] \subseteq N$. Together with (1) this implies

$$[H^*R, H^*R] \subseteq [H^*R, L] \subseteq H^*[R, H^*L] \subseteq H^*[R,L] \subseteq H^*N = N.$$

Thus H^*R is solvable and $H^*R = R$. □

As an immediate consequence, we get D. Pagon, D. Repovš, and M.V. Zaicev's result:

Corollary. *Let L be a finite dimensional Lie algebra over a field F of characteristic 0, graded by an arbitrary group. Then the solvable radical R and the nilpotent radical N of L are graded ideals.*

Combining this with [11, Theorem 4], we obtain the graded Levi theorem:

Corollary. *Let L be a finite dimensional Lie algebra over a field F of characteristic 0, graded by an arbitrary group G. Then there exists a maximal semisimple subalgebra B in L such that $L = B \oplus R$ (direct sum of graded subspaces).*

5. Graded polynomial identities and their codimensions

Let G be a group and let F be a field. Denote by $F\langle X^{\mathrm{gr}}\rangle$ the free G-graded associative algebra over F on the countable set

$$X^{\mathrm{gr}} := \bigcup_{g\in G} X^{(g)},$$

$X^{(g)} = \{x_1^{(g)}, x_2^{(g)}, \dots\}$, i.e. the algebra of polynomials in non-commuting variables from X^{gr}. The indeterminates from $X^{(g)}$ are said to be homogeneous of degree g. The G-degree of a monomial $x_{i_1}^{(g_1)} \dots x_{i_t}^{(g_t)} \in F\langle X^{\mathrm{gr}}\rangle$ is defined to be $g_1 g_2 \dots g_t$, as opposed to its total degree, which is defined to be t. Denote by $F\langle X^{\mathrm{gr}}\rangle^{(g)}$ the subspace of the algebra $F\langle X^{\mathrm{gr}}\rangle$ spanned by all the monomials having G-degree g. Notice that

$$F\langle X^{\mathrm{gr}}\rangle^{(g)} F\langle X^{\mathrm{gr}}\rangle^{(h)} \subseteq F\langle X^{\mathrm{gr}}\rangle^{(gh)},$$

for every $g, h \in G$. It follows that

$$F\langle X^{\mathrm{gr}}\rangle = \bigoplus_{g\in G} F\langle X^{\mathrm{gr}}\rangle^{(g)}$$

is a G-grading. Let $f = f(x_{i_1}^{(g_1)}, \dots, x_{i_t}^{(g_t)}) \in F\langle X^{\mathrm{gr}}\rangle$. We say that f is a *graded polynomial identity* of a G-graded algebra $A = \bigoplus_{g\in G} A^{(g)}$ and write $f \equiv 0$ if $f(a_{i_1}^{(g_1)}, \dots, a_{i_t}^{(g_t)}) = 0$ for all $a_{i_j}^{(g_j)} \in A^{(g_j)}$, $1 \leqslant j \leqslant t$. The set $\mathrm{Id}^{\mathrm{gr}}(A)$ of graded polynomial identities of A is a graded ideal of $F\langle X^{\mathrm{gr}}\rangle$. The case of ordinary polynomial identities is included for the trivial group $G = \{e\}$.

Example 2. Let $G = \mathbb{Z}_2 = \{\bar 0, \bar 1\}$, $M_2(F) = M_2(F)^{(\bar 0)} \oplus M_2(F)^{(\bar 1)}$ where $M_2(F)^{(\bar 0)} = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}$ and $M_2(F)^{(\bar 1)} = \begin{pmatrix} 0 & F \\ F & 0 \end{pmatrix}$. Then $x^{(\bar 0)}y^{(\bar 0)} - y^{(\bar 0)}x^{(\bar 0)} \in \mathrm{Id}^{\mathrm{gr}}(M_2(F))$.

Let $P_n^{\mathrm{gr}} := \langle x^{(g_1)}_{\sigma(1)} x^{(g_2)}_{\sigma(2)} \ldots x^{(g_n)}_{\sigma(n)} \mid g_i \in G, \sigma \in S_n \rangle_F \subset F\langle X^{\mathrm{gr}} \rangle$, $n \in \mathbb{N}$. Then the number

$$c_n^{\mathrm{gr}}(A) := \dim \left(\frac{P_n^{\mathrm{gr}}}{P_n^{\mathrm{gr}} \cap \mathrm{Id}^{\mathrm{gr}}(A)} \right)$$

is called the nth *codimension of graded polynomial identities* or the nth *graded codimension* of A.

The analog of Amitsur's conjecture for graded codimensions can be formulated as follows.

Conjecture. *There exists* $\mathrm{PIexp}^{\mathrm{gr}}(A) := \lim\limits_{n\to\infty} \sqrt[n]{c_n^{\mathrm{gr}}(A)} \in \mathbb{Z}_+$.

Theorem 3. *Let A be a finite dimensional non-nilpotent associative algebra over a field F of characteristic* 0*, graded by any group G. Then there exist constants $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$, $d \in \mathbb{N}$ such that $C_1 n^{r_1} d^n \leqslant c_n^{\mathrm{gr}}(A) \leqslant C_2 n^{r_2} d^n$ for all $n \in \mathbb{N}$.*

Corollary. *The above analog of Amitsur's conjecture holds for such codimensions.*

Remark. If A is nilpotent, i.e. $x_1 \ldots x_p \equiv 0$ for some $p \in \mathbb{N}$, then $P_n^{\mathrm{gr}} \subseteq \mathrm{Id}^{\mathrm{gr}}(A)$ and $c_n^{\mathrm{gr}}(A) = 0$ for all $n \geqslant p$.

Theorem 3 will be proved in Section 7.

6. Polynomial H-identities and their codimensions

In the proof of Theorem 3, we use the notion of generalized Hopf action [6, Section 3].

Let H be an associative algebra with 1 over a field F. We say that an associative algebra A is an algebra with a *generalized H-action* if A is endowed with a homomorphism $H \to \mathrm{End}_F(A)$ and for every $h \in H$ there exist $h'_i, h''_i, h'''_i, h''''_i \in H$ such that

$$h(ab) = \sum_i \bigl((h'_i a)(h''_i b) + (h'''_i b)(h''''_i a) \bigr) \text{ for all } a, b \in A. \tag{3}$$

Example 3. Let A be a *(left) H-module algebra* over a field F where H is a Hopf algebra, i.e. A is endowed with a homomorphism $H \to \mathrm{End}_F(A)$ such that $h(ab) = (h_{(1)}a)(h_{(2)}b)$ for all $h \in H$, $a, b \in A$. Then A is an algebra with a generalized H-action.

Example 4. Let A be a finite dimensional H-comodule algebra over a field F where H is a Hopf algebra. Then by Lemma 1, A is an algebra with a generalized H^*-action.

Example 5. Let $A = \bigoplus_{g\in G} A^{(g)}$ be an algebra graded by a group G and let $(FG)^*$ be the algebra dual to the group coalgebra FG. In other words, $(FG)^*$ is the algebra of functions $G \to F$ with the pointwise multiplication. Then A has the following natural $(FG)^*$-action: $ha^{(g)} = h(g)a^{(g)}$ for all $g \in G$, $a^{(g)} \in A^{(g)}$, and $h \in (FG)^*$. If A is finite dimensional, $A = \bigoplus_{k=1}^m A^{(\gamma_k)}$ for some $m \in \mathbb{Z}_+$, $\gamma_1, \ldots, \gamma_m \in G$. Therefore,

$$h(ab) = \sum_{j,k=1}^m h(\gamma_j\gamma_k)(\pi_{\gamma_j} a)(\pi_{\gamma_k} b) = \sum_{j,k=1}^m h(\gamma_j\gamma_k)(h_{\gamma_j} a)(h_{\gamma_k} b) \text{ for all } h \in (FG)^*,\ a, b \in A,$$

where $h_g \in (FG)^*$ are delta functions: $h_{g_1}(g_2) = \begin{cases} 1, & g_1 = g_2, \\ 0, & g_1 \neq g_2, \end{cases}$ and $\pi_g \colon A \to A^{(g)}$ are the natural projections with the kernels $\bigoplus_{\substack{t\in G,\\ t\neq g}} A^{(t)}$. Hence A is an algebra with a generalized $(FG)^*$-action.

Choose a basis $(\gamma_\beta)_{\beta\in\Lambda}$ in H and denote by $F\langle X|H \rangle$ the free associative algebra over F with free formal generators $x_i^{\gamma_\beta}$, $\beta \in \Lambda$, $i \in \mathbb{N}$. Let $x_i^h := \sum_{\beta\in\Lambda} \alpha_\beta x_i^{\gamma_\beta}$ for $h = \sum_{\beta\in\Lambda} \alpha_\beta \gamma_\beta$, $\alpha_\beta \in F$, where only finite number of α_β are nonzero. Here $X := \{x_1, x_2, x_3, \ldots\}$, $x_j := x_j^1$,

$1 \in H$. We refer to the elements of $F\langle X|H\rangle$ as H-polynomials. Note that here we do not consider any H-action on $F\langle X|H\rangle$.

Let A be an associative algebra with a generalized H-action. Any map $\psi\colon X \to A$ has a unique homomorphic extension $\bar\psi\colon F\langle X|H\rangle \to A$ such that $\bar\psi(x_i^h) = h\psi(x_i)$ for all $i \in \mathbb{N}$ and $h \in H$. An H-polynomial $f \in F\langle X|H\rangle$ is an *H-identity* of A if $\bar\psi(f) = 0$ for all maps $\psi\colon X \to A$. In other words, $f(x_1, x_2, \dots, x_n)$ is an H-identity of A if and only if $f(a_1, a_2, \dots, a_n) = 0$ for any $a_i \in A$. In this case we write $f \equiv 0$. The set $\mathrm{Id}^H(A)$ of all H-identities of A is an ideal of $F\langle X|H\rangle$. Note that our definition of $F\langle X|H\rangle$ depends on the choice of the basis $(\gamma_\beta)_{\beta\in\Lambda}$ in H. However such algebras can be identified in the natural way, and $\mathrm{Id}^H(A)$ is the same.

Denote by P_n^H the space of all multilinear H-polynomials in $x_1, \dots, x_n$, $n \in \mathbb{N}$, i.e.

$$P_n^H = \langle x^{h_1}_{\sigma(1)} x^{h_2}_{\sigma(2)} \dots x^{h_n}_{\sigma(n)} \mid h_i \in H, \sigma \in S_n\rangle_F \subset F\langle X|H\rangle.$$

Then the number $c_n^H(A) := \dim\left(\frac{P_n^H}{P_n^H \cap \mathrm{Id}^H(A)}\right)$ is called the nth *codimension of polynomial H-identities* or the nth *H-codimension* of A.

We need the following theorem:

Theorem 4 ([12, Theorem 1]). *Let A be a finite dimensional non-nilpotent associative algebra with a generalized H-action where H is an associative algebra with 1 over an algebraically closed field F of characteristic 0. Suppose that the Jacobson radical $J(A)$ is an H-submodule and*

$$A/J(A) = B_1 \oplus \ldots \oplus B_q \text{ (direct sum of H-invariant ideals)}$$

for some H-simple algebras B_i. Then there exist constants $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$, and $d \in \mathbb{N}$ such that

$$C_1 n^{r_1} d^n \leqslant c_n^H(A) \leqslant C_2 n^{r_2} d^n \text{ for all } n \in \mathbb{N}.$$

7. Proof of Theorem 3. One example

In order to apply Theorem 4, we need the following lemma:

Lemma 5. *Let $A = \bigoplus_{g\in G} A^{(g)}$ be a finite dimensional associative algebra over a field F graded by any group G. Consider the corresponding generalized H-action for $H = (FG)^*$ (see Example 5). Then $c_n^{\mathrm{gr}}(A) = c_n^H(A)$ for all $n \in \mathbb{N}$.*

Proof. Again, let $\{\gamma_1, \dots, \gamma_m\} := \{g \in G \mid A^{(g)} \ne 0\}$.

Define the homomorphism of algebras $\xi\colon F\langle X|H\rangle \to F\langle X^{\mathrm{gr}}\rangle$ by the formula $\xi(x_k^h) = \sum_{i=1}^m h(\gamma_i) x_k^{(\gamma_i)}$, $h \in H$, $k \in \mathbb{N}$. Note that $\xi(\mathrm{Id}^H(A)) \subseteq \mathrm{Id}^{\mathrm{gr}}(A)$ since for any homomorphism $\psi\colon F\langle X^{\mathrm{gr}}\rangle \to A$ of graded algebras we have $\psi(\xi(x_i^h)) = h\psi(\xi(x_i))$ and if $f \in \mathrm{Id}^H(A)$, then $\psi(\xi(f)) = 0$. Hence we can define $\tilde\xi\colon F\langle X|H\rangle/\mathrm{Id}^H(A) \to F\langle X^{\mathrm{gr}}\rangle/\mathrm{Id}^{\mathrm{gr}}(A)$.

Let $h_{\gamma_1}, \dots, h_{\gamma_m} \in H$ be the delta functions from Example 5. Then $h_{\gamma_i} a \in A^{(\gamma_i)}$ is the γ_i-component of a for all $a \in A$ and $1 \leqslant i \leqslant m$. In particular,

$$x^h - \sum_{i=1}^m h(\gamma_i) x^{h_{\gamma_i}} \in \mathrm{Id}^H(A) \text{ for all } h \in H. \tag{4}$$

We define the homomorphism of algebras $\eta\colon F\langle X^{\mathrm{gr}}\rangle \to F\langle X|H\rangle$ by the formula $\eta(x_j^{(\gamma_i)}) = x_j^{h_{\gamma_i}}$ for all $1 \leqslant i \leqslant m$, $j \in \mathbb{N}$, and $\eta(x_j^{(g)}) = 0$ for $g \notin \{\gamma_1, \dots, \gamma_m\}$. Note that $\eta(\mathrm{Id}^{\mathrm{gr}}(A)) \subseteq \mathrm{Id}^H(A)$. Indeed, if $\psi\colon F\langle X|H\rangle \to A$ is a homomorphism of algebras such that $\psi(x_i^h) = h\psi(x_i)$ for all $h \in H$ and $i \in \mathbb{N}$, then $\psi(\eta(x_j^{(\gamma_i)})) = h_{\gamma_i}\psi(x_j) \in A^{(\gamma_i)}$ for any choice of $\psi(x_j) \in A$. Hence $\psi\eta\colon F\langle X^{\mathrm{gr}}\rangle \to A$ is a graded homomorphism, $\psi(\eta(\mathrm{Id}^{\mathrm{gr}}(A))) = 0$, and $\eta(\mathrm{Id}^{\mathrm{gr}}(A)) \subseteq \mathrm{Id}^H(A)$. Thus we can define $\tilde\eta\colon F\langle X^{\mathrm{gr}}\rangle/\mathrm{Id}^{\mathrm{gr}}(A) \to F\langle X|H\rangle/\mathrm{Id}^H(A)$.

Denote by $\bar f$ the image of a polynomial f in a factor space. Then

$$\tilde\xi\tilde\eta(\bar x_j^{(\gamma_i)}) = \sum_{k=1}^m h_{\gamma_i}(\gamma_k)\bar x_j^{(\gamma_k)} = \bar x_j^{(\gamma_i)} \text{ for all } 1 \leqslant i \leqslant m,\ j \in \mathbb{N}.$$

Since $x_j^{(g)} \in \mathrm{Id}^{\mathrm{gr}}(A)$ for all $g \notin \{\gamma_1, \ldots, \gamma_m\}$, the map $\tilde\xi\tilde\eta$ coincides with the identity map on the generators. Hence $\tilde\xi\tilde\eta = \mathrm{id}_{F\langle X^{\mathrm{gr}}\rangle/\,\mathrm{Id}^{\mathrm{gr}}(A)}$. By (4),

$$\tilde\eta\tilde\xi(\bar x_j^h) = \tilde\eta\left(\sum_{i=1}^m h(\gamma_i)\bar x_j^{(\gamma_i)}\right) = \sum_{i=1}^m h(\gamma_i)\bar x_j^{h_{\gamma_i}} = \bar x_j^h \text{ for all } j \in \mathbb{N},\ h \in H.$$

Similarly, we have $\tilde\eta\tilde\xi = \mathrm{id}_{F\langle X|H\rangle/\,\mathrm{Id}^H}$. Hence

$$F\langle X|H\rangle/\,\mathrm{Id}^H(A) \cong F\langle X^{\mathrm{gr}}\rangle/\,\mathrm{Id}^{\mathrm{gr}}(A).$$

In particular, $\frac{P_n^H}{P_n^H \cap \mathrm{Id}^H(A)} \cong \frac{P_n^{\mathrm{gr}}}{P_n^{\mathrm{gr}} \cap \mathrm{Id}^{\mathrm{gr}}(A)}$ and $c_n^{\mathrm{gr}}(A) = c^H(A)$ for all $n \in \mathbb{N}$. ☐

Proof of Theorem 3. By Example 5, A is an algebra with a generalized H-action for $H = (FG)^*$. By Lemma 5, graded codimensions of A coincide with its H-codimensions. By Theorem 1, $J(A)$ is a graded ideal and, therefore, an H-submodule. By Lemma 3, $A/J(A)$ is a direct sum of graded simple algebras. Now we use Theorem 4. ☐

Example 6. Let G be the free group with free generators $a_1, \ldots, a_\ell$, $\ell \geqslant 2$, let F be a field of characteristic 0, and let $k \in \mathbb{N}$. Recall that the group algebra FG has the natural G-grading

$$FG = \bigoplus_{g \in G} (FG)^{(g)} \text{ where } (FG)^{(g)} = Fg,\ g \in G.$$

Consider the subalgebra A of FG generated by $1, a_1, \ldots, a_\ell$ and the ideal $I_k \subset A$ generated by all products $a_{i_1} \ldots a_{i_n}$ of length $n \geqslant k$. Note that both A and I_k are graded and A/I_k is a finite dimensional algebra graded by an infinite non-Abelian group G. We claim that there exist $r_1, r_2 \in \mathbb{R}$, $C_1, C_2 > 0$ such that $C_1 n^{r_1} \leqslant c_n^{\mathrm{gr}}(A/I_k) \leqslant C_2 n^{r_2}$ for all $n \in \mathbb{N}$.

Proof. Note that $A/I_k = (F1) \oplus J$ (direct sum of subspaces) where J is the ideal generated by the images of a_i in A/I_k. Moreover, J is the Jacobson radical of A/I_k since J is nilpotent. Using Theorem 3, Lemma 5, and the formula in [12, Theorem 1], we get

$$\mathrm{PIexp}^{\mathrm{gr}}(A/I_k) = \dim((A/I_k)/J) = 1.$$

☐

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References

[1] Abe, E. Hopf algebras. Cambridge University Press, Cambridge, 1980.
[2] Aljadeff, E., Giambruno, A. Multialternating graded polynomials and growth of polynomial identities. *Proc. Amer. Math. Soc.* (to appear).
[3] Aljadeff, E., Giambruno, A., La Mattina, D. Graded polynomial identities and exponential growth. *J. reine angew. Math.*, **650** (2011), 83–100.
[4] Bahturin, Yu. A., Giambruno, A., Zaicev, M. V. G-identities on associative algebras. *Proc. Amer. Math. Soc.*, **127**:1 (1999), 63–69.
[5] Bahturin, Yu. A., Zaicev, M. V. Identities of graded algebras and codimension growth. *Trans. Amer. Math. Soc.* **356**:10 (2004), 3939–3950.

[6] Berele, A. Cocharacter sequences for algebras with Hopf algebra actions. *J. Algebra*, **185** (1996), 869–885.

[7] Cohen, M., Montgomery, S. Group graded rings, smash products, and group actions, *Trans. Amer. Math. Soc.*, **282** (1984) 237–258; Addendum, *Trans. Amer. Math. Soc.* **300** (1987) 810–811.

[8] Dăscălescu, S., Năstăsescu, C., Raianu, Ş. Hopf algebras: an introduction. New York, Marcel Dekker, Inc., 2001.

[9] Giambruno, A., La Mattina, D. Graded polynomial identities and codimensions: computing the exponential growth. *Adv. Math.*, **225** (2010), 859–881.

[10] Giambruno, A., Zaicev, M. V. Polynomial identities and asymptotic methods. AMS Mathematical Surveys and Monographs Vol. 122, Providence, R.I., 2005.

[11] Gordienko, A. S. Structure of H-(co)module Lie algebras. *J. Lie Theory* (to appear)

[12] Gordienko, A. S. Asymptotics of H-identities for associative algebras with an H-invariant radical. `arXiv:1212.1321 [math.RA] 6 Dec 2012`

[13] Gordienko, A. S., Kotchetov, M. V. Derivations, gradings, actions of algebraic groups, and codimension growth of polynomial identities. `arXiv:1210.2528 [math.RA] 9 Oct 2012`

[14] Goto, M., Grosshans, F. Semisimple Lie algebras. Marcel Dekker, New York and Basel, 1978.

[15] Linchenko, V. Nilpotent subsets of Hopf module algebras, *Groups, rings, Lie, and Hopf algebras*, Proc. 2001 St. John's Conference, ed. Yu. Bahturin (Kluwer, 2003) 121–127.

[16] Linchenko, V., Montgomery, S., Small, L. W. Stable Jacobson radicals and semiprime smash products. *Bull. London Math. Soc.*, **37** (2005), 860–872.

[17] Montgomery, S. Hopf algebras and their actions on rings, CBMS Lecture Notes **82**, Amer. Math. Soc., Providence, RI, 1993.

[18] Pagon, D., Repovš, D., Zaicev, M. V. Group gradings on finite dimensional Lie algebras, *Alg. Colloq.* (to appear).

[19] Skryabin, S. M. Structure of H-semiprime Artinian algebras. *Algebra and Representation Theory*, **14** (2011), 803–822.

[20] Ştefan, D., Van Oystaeyen, F. The Wedderburn — Malcev theorem for comodule algebras. *Comm. in Algebra*, **27**:8 (1999), 3569–3581.

[21] Sweedler, M. Hopf algebras. W.A. Benjamin, inc., New York, 1969.

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